

ACTION INTEGRALS AND INFINITESIMAL CHARACTERS

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ABSTRACT. Let G be a reductive Lie group and \mathcal{O} the coadjoint orbit of a hyperbolic element of \mathfrak{g}^* . By π is denoted the unitary irreducible representation of G associated with \mathcal{O} by the orbit method. We give geometric interpretations in terms of concepts related to \mathcal{O} of the constant $\pi(g)$, for $g \in Z(G)$. We also offer a description of the invariant $\pi(g)$ in terms of action integrals and Berry phases. In the spirit of the orbit method we interpret geometrically the infinitesimal character of the differential representation of π .

MSC 2000: Primary: 53D50, Secondary: 22E45

1. INTRODUCTION

Roughly speaking, the orbit method [3], [14] suggests that the unitary dual of a Lie group G (i.e. the set of equivalence classes of unitary irreducible representations of G) is in bijective correspondence with the space of coadjoint orbits of G . Moreover the orbit method relates geometric properties of the coadjoint orbit with properties of the corresponding irreducible representation. This bijective correspondence exists if G is a connected simply connected nilpotent group; in other cases where the correspondence is not a perfect bijection this method gives valuable suggestions about the geometric meaning of some facts of representation theory.

In this paper G will be a reductive group and \mathcal{O} will be a coadjoint orbit of a hyperbolic element $\eta \in \mathfrak{g}^*$, where \mathfrak{g}^* is the dual of the Lie algebra \mathfrak{g} of G . In the spirit of the orbit method we will give geometric interpretations of some invariants of the representation associated with \mathcal{O} . This will allow us, in turn, to offer physical interpretations of those invariants in terms of action integrals and Berry phases along curves generated in physical systems by the action of symmetry groups. This is valid for groups relevant in Physics, such as: $SO(p, q)$, $Sp(2n)$, $SL(n, \mathbb{R})$, etc.

For the construction of a representation of G from the orbit we will assume that \mathcal{O} admits an integral datum (see Section 2). By means of an integral datum one defines a unitary irreducible representation π of G by induction from a parabolic subgroup of G . According to Schur's lemma, if g_1 belongs to the center of G , $\pi(g_1)$ is a scalar operator defined by a constant κ ,

$$(1.1) \quad \pi(g_1) = \kappa \text{Id}.$$

We will give an interpretation in geometric terms of:

- 1) The constant κ .

Key words and phrases. Orbit method, geometric quantization, coadjoint orbits, representation theory.

This work has been partially supported by Ministerio de Educación y Ciencia, grant MAT2007-65097-C02-02.

2) The infinitesimal character [5] of π' , the differential representation of π , considered as a representation of the universal enveloping algebra $U(\mathfrak{g})$ of \mathfrak{g} .

3) Some values of the character χ_τ of τ , where τ is any irreducible representation of any maximal compact subgroup K of G , which occurs in $\pi|_K$.

The orbit \mathcal{O} is the homogeneous space G/L , where L is the stabilizer of η . An integral datum is a unitary irreducible representation of L on a Hilbert space H , satisfying an additional condition (see (2.5)). Given an integral datum Λ , by Φ we denote the representation of L , tensor product of Λ and the character of L on half-densities (2.3).

As η is a hyperbolic element its orbit \mathcal{O} possesses a real polarization defined by a subalgebra \mathfrak{u} of \mathfrak{g} (see (2.2)). \mathcal{B}_1 will be the space of smooth Φ -equivariant maps $s : G \rightarrow H$, with compact support modulo L , and such that $L_C(s) = 0$, for $C \in \mathfrak{u}$, where L_A is the left invariant vector field on G determined by $A \in \mathfrak{g}$. The representation π is the left regular representation of G defined on the completion of the pre-Hilbert space \mathcal{B}_1 . Thus the operator associated to $A \in \mathfrak{g}$ in the differential representation π' of π is $-R_A$, the opposite of the right invariant vector field on G defined by A .

As a first step we will define the representation π' in the context of fibre bundles. We will consider the $GL(H)$ -principal bundle $\mathcal{F} := G \times_\Phi GL(H)$ over \mathcal{O} , defined by means of the representation Φ of L . On \mathcal{F} there is a natural left G -action and an obvious \mathbb{C}^* -action induced by the multiplication by nonzero scalars of elements of $GL(H)$. In particular each $A \in \mathfrak{g}$ defines a vector field Y_A on \mathcal{F} by the G -action. Furthermore, on \mathcal{F} one can define a G -invariant connection in a natural way, whose curvature is denoted by \mathbf{K} . The G -action on \mathcal{F} has a moment map $\mu : \mathcal{F} \rightarrow \mathfrak{gl}(H) \otimes \mathfrak{g}^*$, relative to the 2-form \mathbf{K} ; that is, $D\langle \mu, A \rangle = -\mathbf{K}(Y_A, \cdot)$, where D is the covariant derivative. Moreover $\langle \mu, A \rangle =: \mathbf{h}_A$ induces a map h_A from \mathcal{O} to $\mathfrak{gl}(H)$.

We will denote by \mathcal{V} the vector bundle with fibre H associated to \mathcal{F} . We write \mathcal{B}_2 for the space of smooth sections σ of \mathcal{V} which can be identified with the elements of \mathcal{B}_1 , and we put \mathcal{B}_3 for the space of maps $f : \mathcal{F} \rightarrow H$ associated with the sections of \mathcal{V} that belong to \mathcal{B}_2 .

Given $A \in \mathfrak{g}$ we will denote by X_A the vector field on \mathcal{O} defined by the coadjoint action of G . On the space \mathcal{B}_2 we consider the following operator

$$\mathcal{P}_A := -D_{X_A} + h_A.$$

In Section 3 we prove the following theorem, that gives the representation π' on the spaces \mathcal{B}_i , for $i = 2, 3$.

Theorem 1. *The representations of \mathfrak{g}*

$$A \in \mathfrak{g} \mapsto \mathcal{P}_A \in \text{End}(\mathcal{B}_2)$$

and

$$A \in \mathfrak{g} \mapsto -Y_A \in \text{End}(\mathcal{B}_3)$$

are equivalent to π' , the differential representation of π .

Theorem 1 gives the representation π' on geometric objects. To determine a geometric description of $\pi(g_1)$ we will “integrate” π' along a curve in G with final point at g_1 . To abbreviate, the smooth curves in G with initial point at e will be called *paths* in G . Let $\{g_t\}$ be a path in G with g_1 in $Z(G)$, the center of G . This curve determines its velocity curve; that is, the family $\{A_t\} \subset \mathfrak{g}$ given by the

relation $\dot{g}_t g_t^{-1} = A_t$. The corresponding time-dependent vector field Y_{A_t} defines a Hamiltonian flow \mathbf{F}_t on \mathcal{F} . We will prove that the time-1 map of this flow is precisely the multiplication by κ in \mathcal{F} (see item (d) of Theorem 2).

Given $s \in \mathcal{B}_1$, we define a family of maps $\{s_t : G \rightarrow H\}_t$ by the equations

$$(1.2) \quad \frac{ds_t}{dt} = -R_{A_t} s_t, \quad s_0 = s.$$

Given a section $\sigma \in \mathcal{B}_2$ we consider the family of sections σ_t of \mathcal{V} determined by the following equations

$$(1.3) \quad \frac{d\sigma_t}{dt} = \mathcal{P}_{A_t} \sigma_t, \quad \sigma_0 = \sigma.$$

Similarly, given $f \in \mathcal{B}_3$, let $\{f_t\}$ be the set of maps $f_t : \mathcal{F} \rightarrow H$ such that

$$(1.4) \quad \frac{df_t(p)}{dt} = -Y_{A_t}(p)(f_t), \quad f_0 = f.$$

The following theorem relates the constant κ with the solutions of the “evolution” equations (1.2), (1.3), (1.4) and with the time-1 map \mathbf{F}_1 .

Theorem 2. *Let g_t be an arbitrary path on G with $g_1 \in Z(G)$, and A_t the corresponding velocity curve. If κ is the constant given by (1.1), the following statements hold*

- (a) *If s_t is the solution of (1.2), then $s_1 = \kappa s$.*
- (b) *If σ_t is the solution to (1.3), then $\sigma_1 = \kappa \sigma$.*
- (c) *If f_t is the solution to (1.4), then $f_1 = \kappa f$.*
- (d) *\mathbf{F}_1 is the multiplication by κ ; that is, $\mathbf{F}_1[g, \alpha] = \kappa[g, \alpha]$.*

For the above path $\{g_t\}$ with endpoint at $g_1 \in Z(G)$, we denote by ψ_t the closed isotopy on \mathcal{O} determined by the time-dependent vector field X_{A_t} , that is,

$$(1.5) \quad \frac{d\psi_t}{dt} = X_{A_t} \circ \psi_t, \quad \psi_0 = \text{id}.$$

When $\dim H = 1$ the curvature \mathbf{K} projects a 2-form $-\omega$ on the orbit \mathcal{O} , and \mathbf{F}_1 is the action integral ([15], [9]) around ψ_t (see Section 5). This fact is the statement of the following theorem, which will be proved in Section 4

Theorem 3. *If $\dim H = 1$*

$$\kappa = \exp\left(\int_S \omega + \int_0^1 h_{A_t}(q_t) dt\right),$$

where q is an arbitrary point of \mathcal{O} and S any 2-chain in \mathcal{O} whose boundary is the curve $\{q_t := \psi_t(q)\}_t$.

Let K be a maximal compact subgroup of G , and let \tilde{g} an element of K , such that its conjugacy class meets L_0 , the connected component of the identity of L ; that is, there exists $a \in G$ such that $a^{-1}\tilde{g}a \in L_0$. Let g_t be a path in G with $g_1 = \tilde{g}$ and $a^{-1}g_t a \in L$, A_t the corresponding velocity path and h_{A_t} the map on \mathcal{O} induced by $\langle \mu, A_t \rangle$. Let τ be an irreducible representation of K which occurs in the representation $\pi|_K$ with nonzero multiplicity. We will prove the following theorem, that gives an expression for the value of character χ_τ at \tilde{g} in terms of the “Hamiltonian” functions h_{A_t} .

Theorem 4. *Let \tilde{g} be an element of K , such that there exists $a \in G$ with $a^{-1}\tilde{g}a$ in L_0 . Let C_t be the velocity curve of a path in L with endpoint at $a^{-1}\tilde{g}a$. If $\dim \tau = m$ and $\dim H = 1$, then*

$$\chi_\tau(\tilde{g}) = m \exp\left(\int_0^1 h_{A_t}(x_0) dt\right),$$

where $A_t = \text{Ad}_a C_t$ and $x_0 = a \cdot \eta \in \mathcal{O}$.

An important invariant of the representation π' is its infinitesimal character χ defined on $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$, the center of the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ [5]. If \mathfrak{h} is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ contained in $\mathfrak{l}_{\mathbb{C}}$ and $Z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$, we will denote by \hat{Z} the projection of Z into $U(\mathfrak{h})$. Let Y_1, \dots, Y_r be a basis of \mathfrak{h} . For each Y_i we define the map $\mathbf{h}_i : \mathcal{F} \rightarrow \mathfrak{gl}(H)$, by $\mathbf{h}_i = \langle \mu, Y_i \rangle$; that is, \mathbf{h}_i is the Hamiltonian function associated with Y_i . If $q(Y_1, \dots, Y_r)$ is a polynomial

$$(1.6) \quad q(Y_1, \dots, Y_r) = a + \sum_k a_k Y_k + \sum_{i,j} a_{ij} Y_i Y_j + \sum_{k,i,j} a_{kij} Y_k Y_i Y_j + \dots \quad (\text{finite sum})$$

such that $\hat{Z} = q(Y_1, \dots, Y_r) \in U(\mathfrak{h})$, a geometric interpretation of $\chi(Z)$ is given in the following theorem

Theorem 5. *If Z is an element of $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ such that \hat{Z} is defined by the polynomial $q(Y_1, \dots, Y_r)$. Then*

$$q(\mathbf{h}_1, \dots, \mathbf{h}_r) : \mathcal{F} \rightarrow \mathfrak{gl}(H)$$

is a constant map on the fiber over η , and its value on this fiber is $\chi(Z)\text{Id}$.

This article is organized as follows. In Section 2 we introduce the definitions and notations which will be used. Following Vogan ([13], [14]) we define the representation π associated to the orbit \mathcal{O} of a hyperbolic element.

In Section 3 we describe the differential representation π' on the spaces \mathcal{B}_i , for $i = 2, 3$, proving Theorem 1.

In Section 4 we give geometric interpretations of $\pi(g_1)$, for $g_1 \in Z(G)$. We will prove Theorem 2 and Theorem 3. In Subsection 4.2 we prove Theorem 4 about the character of π . Subsection 4.3 concerns with the geometric interpretation of the infinitesimal character of π' ; in this subsection we will prove Theorem 5.

Section 5 provides an interpretation of Theorem 3 in terms of physical concepts. We will show that the invariant κ can be considered as the exponential of the action integral around the closed curve ψ_t , and also as the Berry phase of a loop of Lagrangian submanifolds of \mathcal{O} . In a worked example we will consider a hyperbolic orbit of the restricted Lorentz group $SO^+(1, 3)$. Using Theorem 5 we will calculate the value of the corresponding infinitesimal character on the Casimir element C , and we will interpret this value in terms of the “quantum” operator that Geometric Quantization associates with C .

2. DEFINITIONS AND NOTATIONS.

Here we review the construction of the representation associated to the coadjoint orbit of a hyperbolic element (see [13], [14] for details).

By G we denote a reductive group. As definition of reductive group we adopt the one given by Vogan in [14]. We recall this definition. A *linear* group is reductive if it has finitely many connected components and is preserved by the Cartan involution. A reductive group G is a Lie group endowed with a homomorphism

from G onto G_1 of finite kernel, G_1 being a linear reductive group. In particular, $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, $SO(p, q)$, $O(p, q)$, $Sp(2n)$ and all compact Lie groups are reductive groups. If $g \in G$ and $A \in \mathfrak{g}$, we put $g \cdot A = \text{Ad}_g(A)$, and if $\xi \in \mathfrak{g}^*$ we write $g \cdot \xi$ for $\text{Ad}_g^*(\xi)$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of \mathfrak{g} , with \mathfrak{k} the Lie algebra of a maximal compact subgroup K of G .

Given $\eta \in \mathfrak{g}^*$, its stabilizer for the coadjoint adjoint of G will be denoted by L . On the other hand, η determine an element $X_0 \in \mathfrak{g}$ by the equality

$$(2.1) \quad \eta(Y) = \text{Re Tr}(X_0 Y), \quad \text{for all } Y \in \mathfrak{g},$$

where \mathfrak{g} is identified with a Lie algebra of matrices.

Let us assume that η is a *hyperbolic* element of \mathfrak{g}^* . We can suppose that $X_0 \in \mathfrak{p}$, after replacing η by an $\text{Ad}^*(G)$ -equivariant element. As $\text{ad}(X_0)$ is a diagonalizable endomorphism of \mathfrak{g} with real eigenvalues

$$\mathfrak{g} = \bigoplus_{r \in \mathbb{R}} \mathfrak{g}_r, \quad \mathfrak{g}_r = \{Y \in \mathfrak{g} \mid \text{ad}(X_0)(Y) = rY\},$$

and \mathfrak{g}_0 is the Lie algebra \mathfrak{l} of the subgroup L . Moreover the adjoint action of $l \in L$ preserves each \mathfrak{g}_r .

We put

$$(2.2) \quad \mathfrak{u} = \bigoplus_{r>0} \mathfrak{g}_r, \quad \mathfrak{u}^- = \bigoplus_{r<0} \mathfrak{g}_r, \quad U = \exp \mathfrak{u}.$$

Then U is a simply connected nilpotent subgroup of G and L normalizes U . So $Q := LU$ is a Levi decomposition of the subgroup Q .

We define the following positive character on Q

$$(2.3) \quad \Delta(q) = |\det(\text{Ad}(q)|_{\mathfrak{u}})|^{1/2}.$$

The derivative of Δ will be denoted by δ . Since $[\mathfrak{g}_r, \mathfrak{g}_s] \subset \mathfrak{g}_{r+s}$, $\delta(B) = 0$ for all $B \in \mathfrak{u}$. On the other hand $\Delta(l l_1 l^{-1}) = \Delta(l_1)$, for all $l, l_1 \in L$, so $\delta(l \cdot A) = \delta(A)$, for $l \in L$ and $A \in \mathfrak{l}$. We extend δ to a linear map on \mathfrak{g} by setting $\delta|_{\mathfrak{u}^-} = 0$. As the action of L preserves the \mathfrak{g}_r , then

$$(2.4) \quad \delta(l \cdot Y) = \delta(Y), \quad \text{for all } l \in L \text{ and } Y \in \mathfrak{g}.$$

Let Λ be an *integral datum* at η [13]; that is, Λ is an irreducible unitary representation of L in a Hilbert space H , such that

$$(2.5) \quad \Lambda(\exp A) = e^{i\eta(A)} \text{Id}, \quad \text{for all } A \in \mathfrak{l}.$$

So $\Lambda(l \exp(A) l^{-1}) = \Lambda(\exp A)$. We extend Λ to Q by $\Lambda(lu) = \Lambda(l)$, and write λ for the derivative of Λ . In turn, λ can be extended to a linear map on \mathfrak{g} by putting $\lambda|_{\mathfrak{u}^-} = 0$. As in the preceding case

$$(2.6) \quad \lambda(l \cdot Y) = \lambda(Y), \quad \text{for all } l \in L \text{ and } Y \in \mathfrak{g}.$$

$\text{GL}(H)$ will denote the group of continuous linear automorphisms of H . We put $\Phi : Q \rightarrow \text{GL}(H)$ for the representation of Q tensor product $\Lambda \otimes \Delta$. By ϕ we denote the linear map

$$(2.7) \quad \phi : A \in \mathfrak{g} \mapsto \lambda(A) + \delta(A) \text{Id} \in \mathfrak{gl}(H).$$

From (2.4) and (2.6) it follows

$$(2.8) \quad l \cdot \phi = \phi.$$

By π we denote the representation $\text{Ind}_Q^G(\Phi)$; that is, the irreducible unitary representation of G induced by Φ . The space of C^∞ vectors of π is the space of smooth functions $s : G \rightarrow H$, with compact support modulo L , such that

$$(2.9) \quad s(gl) = \Phi(l^{-1})s(g), \text{ for all } g \in G, l \in L; \quad \text{and } L_A s = 0, \text{ for all } A \in \mathfrak{u},$$

where L_A is the left invariant vector field on G defined by A . The representation π on this space is given by

$$(2.10) \quad \pi(g)(s) = s \circ L_{g^{-1}},$$

$L_{g^{-1}}$ being the left multiplication in G by g^{-1} . Therefore the differential representation π' of π on the smooth function s , with compact support modulo L , which satisfy (2.9) is given by

$$(2.11) \quad \pi'(C)(s) = -R_C(s),$$

R_C being the right invariant vector field on G defined by $C \in \mathfrak{g}$.

3. THE DIFFERENTIAL REPRESENTATION

Henceforth G will be a reductive group, η a hyperbolic element of \mathfrak{g}^* and Λ an integral datum at η . The coadjoint orbit of η will be denoted by \mathcal{O} . If $B \in \mathfrak{g}$, $X_B(g \cdot \eta)$ will be the tangent vector to \mathcal{O} at $g \cdot \eta$ defined by the curve $\{\exp(tB) \cdot (g \cdot \eta)\}_t$.

By (2.8) the map

$$(3.1) \quad h_B : g \in G \mapsto \phi(g^{-1} \cdot B) \in \mathfrak{gl}(H)$$

induces a mapping on \mathcal{O} , that will be also denoted by h_B . For any $A, B \in \mathfrak{g}$

$$(3.2) \quad X_A(h_B) = -h_{[A, B]}.$$

Next we define the following $GL(H)$ -principal bundle over $\mathcal{O} \simeq G/L$.

$$\mathcal{F} = G \times_L GL(H) = \{(g, \alpha) \mid g \in G, \alpha \in GL(H)\} / \sim,$$

where $(g, \alpha) \sim (gl, \Phi(l^{-1})\alpha)$, with $l \in L$.

On \mathcal{F} there is a natural left G -action \mathcal{L} , so each $B \in \mathfrak{g}$ determines a vector field Y_B on \mathcal{F} and

$$(\mathcal{L}_g)_*(Y_B) = Y_{g \cdot B}.$$

On the other hand, each $y \in \mathfrak{gl}(H)$ determines a vertical vector field W_y on \mathcal{F} by means of right $GL(H)$ -action \mathcal{R} .

Given $A \in \mathfrak{l}$, the trivial curve $\{[g \cdot e^{tA}, \Phi(e^{-tA})\alpha]\}_t$ in \mathcal{F} defines the vector $Y_{g \cdot A}[g, \alpha] - W_y[g, \alpha]$, with $y = \text{Ad } \alpha^{-1}(\phi(A))$, where Ad denotes the adjoint action of $GL(H)$ on $\mathfrak{gl}(H)$. So the tangent space to \mathcal{F} at $[g, \alpha]$ is

$$(3.3) \quad T_{[g, \alpha]} \mathcal{F} = \frac{\{Y_B[g, \alpha] \mid B \in \mathfrak{g}\} \oplus \{W_y[g, \alpha] \mid y \in \mathfrak{gl}(H)\}}{\{Y_{g \cdot A}[g, \alpha] - W_{\text{Ad } \alpha^{-1}(\phi(A))}[g, \alpha] \mid A \in \mathfrak{l}\}}$$

On \mathcal{F} we define the following $\mathfrak{gl}(H)$ -valued 1-form

$$(3.4) \quad \Omega(Y_B[g, \alpha] + W_y[g, \alpha]) = \text{Ad } \alpha^{-1}(\phi(g^{-1} \cdot B)) + y.$$

Ω is, in fact, well-defined on the quotient (3.3), and it is easy to check that

$$\mathcal{L}_g^* \Omega = \Omega, \quad \text{and} \quad \mathcal{R}_\alpha^* \Omega = \text{Ad}_{\alpha^{-1}} \circ \Omega,$$

for $g \in G$ and $\alpha \in GL(H)$. That is, one has the proposition

Proposition 6. *The 1-form Ω defined in (3.4) is a G -invariant connection on the $GL(H)$ -principal bundle \mathcal{F} .*

We can lift h_A to a well-defined function $\mathbf{h}_A : \mathcal{F} \rightarrow \mathfrak{gl}(H)$ by setting

$$(3.5) \quad \mathbf{h}_A[g, \alpha] = \text{Ad}_{\alpha^{-1}} h_A(g).$$

Then

$$(3.6) \quad \Omega(Y_B + W_y) = \mathbf{h}_B + y$$

Lemma 7. *Given $A \in \mathfrak{g}$ and $y \in \mathfrak{gl}(H)$, then*

$$Y_A(\mathbf{h}_B) = -\mathbf{h}_{[A, B]} \quad \text{and} \quad W_y[g, \alpha](\mathbf{h}_B) = -[y, \mathbf{h}_B([g, \alpha])]_{\mathfrak{gl}},$$

where $[\cdot, \cdot]_{\mathfrak{gl}}$ is the bracket in the Lie algebra $\mathfrak{gl}(H)$.

Proof. From (3.2) it follows

$$Y_A[g, \alpha](\mathbf{h}_B) = \frac{d}{dt} \Big|_{t=0} \mathbf{h}_B[\exp(tA) \cdot g, \alpha] = \text{Ad}_{\alpha^{-1}}(X_A(g)(h_B)) = -\mathbf{h}_{[A, B]}[g, \alpha].$$

The second formula can be directly deduced from

$$\mathbf{h}_B[g, \alpha \exp(ty)] = \text{Ad}_{\exp(-ty)} \mathbf{h}_B[g, \alpha].$$

□

Next we will calculate the value of the curvature \mathbf{K} of the connection Ω on the pair (Y_B, Y_C) of vector fields.

Proposition 8. *The curvature \mathbf{K} of the connection Ω satisfies*

$$(3.7) \quad \mathbf{K}(Y_B, Y_C) = -\mathbf{h}_{[B, C]} + [\mathbf{h}_B, \mathbf{h}_C]_{\mathfrak{gl}},$$

for all $B, C \in \mathfrak{g}$.

Proof. By the structure equation

$$\mathbf{K}(Y_B, Y_C) = d\Omega(Y_B, Y_C) + [\Omega(Y_B), \Omega(Y_C)]_{\mathfrak{gl}}.$$

From Lemma 7 and (3.6) it follows

$$Y_B[g, \alpha](\Omega(Y_C)) = \mathbf{h}_{[C, B]}[g, \alpha].$$

Similarly $Y_C(\Omega(Y_B)) = \mathbf{h}_{[B, C]}$. Hence $d\Omega(Y_B, Y_C) = -\mathbf{h}_{[B, C]}$ and (3.7) follows. □

By D we denote the covariant derivative determined by Ω . Since the horizontal component of $Y_A[g, \alpha]$ is $Y_A[g, \alpha] + W_y[g, \alpha]$, with $y = -\mathbf{h}_A[g, \alpha]$, by Lemma 7

$$D\mathbf{h}_B(Y_A[g, \alpha]) = -\mathbf{h}_{[A, B]} + [\mathbf{h}_A, \mathbf{h}_B]_{\mathfrak{gl}}$$

It follows from (3.7) that

$$D\mathbf{h}_B(Y_A) = -\mathbf{K}(Y_B, Y_A), \quad \text{for all } A \in \mathfrak{g}.$$

Thus we have

$$(3.8) \quad D\mathbf{h}_B = -\iota_{Y_B} \mathbf{K}, \quad \text{for all } B \in \mathfrak{g}.$$

Equation (3.8) can be interpreted saying that the $\mathfrak{gl}(H)$ -valued function \mathbf{h}_B is a “Hamiltonian” for the vector field Y_B , with respect to the covariantly closed $\mathfrak{gl}(H)$ -valued 2-form \mathbf{K} . That is, we define

$$\mu : \mathcal{F} \rightarrow \mathfrak{gl}(H) \otimes \mathfrak{g}^*$$

by $\langle \mu, A \rangle = \mathbf{h}_A$, with $A \in \mathfrak{g}$. This map is G -equivariant, that is, $\langle \mu(gp), A \rangle = \langle \mu(p), g^{-1} \cdot A \rangle$, for all $p \in \mathcal{F}$ and all $g \in G$ and $D\langle \mu, A \rangle = -\iota_{Y_B} \mathbf{K}$. We call μ the moment map for the G -action on $(\mathcal{F}, \mathbf{K})$.

We denote by \mathcal{V} the vector bundle on \mathcal{O} with fibre H

$$G \times_L H = \{\langle g, v \rangle \mid g \in G, v \in H\},$$

with $\langle g, v \rangle = \langle gl, \Phi(l^{-1})v \rangle$. The vector bundle \mathcal{V} can also be considered as associated to \mathcal{F} by the natural representation of $GL(H)$. That is, \mathcal{V} is

$$\{\{p, v\} \mid p \in \mathcal{F}, v \in H\},$$

where $\{p, v\} = \{p\beta, \beta^{-1}v\}$ for all $\beta \in GL(H)$. The correspondence $\langle g, v \rangle \mapsto \{[g, \text{Id}], v\}$ gives the isomorphism between those definitions of \mathcal{V} .

Now we consider the following three vector spaces of *smooth* maps

$$\mathcal{B}'_1 = \{s : G \rightarrow H \mid s(gl) = \Phi(l^{-1})s(g), \forall g \in G, \forall l \in L, \text{supp}(s) \text{ compact modulo } L\}$$

$$\mathcal{B}'_2 = \{\tau \mid \tau \text{ section of } \mathcal{V}, \text{supp}(\tau) \text{ compact}\}$$

$$\mathcal{B}'_3 = \{f : \mathcal{F} \rightarrow H \mid f(p\beta) = \beta^{-1}f(p), \forall p \in \mathcal{F}, \forall \beta \in GL(H), \text{pr}(\text{supp}(f)) \text{ compact}\},$$

where pr is the projection $\text{pr} : \mathcal{F} \rightarrow G/L$.

Given $s \in \mathcal{B}'_1$, determines a section $\sigma \in \mathcal{B}'_2$ by the relation

$$(3.9) \quad \sigma(gL) = \langle g, s(g) \rangle.$$

Moreover s defines $\sigma^\# \in \mathcal{B}'_3$ by

$$(3.10) \quad \sigma^\#[g, \alpha] = \alpha^{-1}s(g).$$

With the above notations

$$(3.11) \quad \sigma(x) = \{p, \sigma^\#(p)\},$$

for any $p \in \mathcal{F}$ in the fibre of $x \in \mathcal{O}$.

It is well-known that the correspondences $s \mapsto \sigma$ and $s \mapsto \sigma^\#$ allow us to identify the \mathcal{B}'_i 's. We denote also by D the covariant derivative on sections of \mathcal{V} defined by the connection Ω . It is a known fact that the map of \mathcal{B}'_3 associated with the section $D_{X_A}\sigma$ of \mathcal{V} is $X_A^\#(\sigma^\#)$, where $X_A^\#$ is the horizontal lifting of the vector field X_A .

We set

$$(3.12) \quad \mathcal{B}_1 = \{s \in \mathcal{B}'_1 \mid L_A s = 0, \forall A \in \mathfrak{u}\}.$$

Since Q is connected the condition defining \mathcal{B}_1 is equivalent to $s(qq) = \Phi(q^{-1})s(g)$ for all $q \in Q$ and all $g \in G$. In order to interpret \mathcal{B}_1 in terms of sections of \mathcal{V} and equivariant functions on \mathcal{F} we need an additional fibre bundle.

\mathcal{F}_Q is the $GL(H)$ -principal fibre bundle over G/Q defined by Φ ; that is,

$$\mathcal{F}_Q := G \times_Q GL(H).$$

One has a natural fibre map $\Xi : \mathcal{F} \rightarrow \mathcal{F}_Q$ over the canonical projection

$$(3.13) \quad G/L \rightarrow G/Q.$$

We put

$$(3.14) \quad \mathcal{B}_3 = \{f \in \mathcal{B}'_3 \mid f \text{ factors through } \Xi\}.$$

Analogously we define $\mathcal{V}_Q := G \times_Q H$, and the natural fibre map $\Xi : \mathcal{V} \rightarrow \mathcal{V}_Q$ will be also denoted by Ξ . We set

$$(3.15) \quad \mathcal{B}_2 := \{\tau \in \mathcal{B}'_2 \mid \Xi \circ \sigma \text{ is constant along the fibers of } (3.13)\}.$$

From the above definitions is easy to prove the following proposition

Proposition 9. *The correspondences $s \mapsto \sigma$ and $s \mapsto \sigma^\sharp$ define bijective maps between the \mathcal{B}_i 's.*

Proof of Theorem 1. From (3.10) one deduces

$$(3.16) \quad Y_A[g, 1](\sigma^\sharp) = R_A(g)(s), \quad \text{and} \quad W_y[g, 1](\sigma^\sharp) = -y s(g).$$

On the other hand, by (3.6)

$$(3.17) \quad X_B^\sharp[g, \alpha] = Y_B[g, \alpha] + W_y[g, \alpha],$$

with $y + \mathbf{h}_B[g, \alpha] = 0$.

From (3.16) together with (3.17) and (3.10) it follows that the Φ -equivariant function on G associated with $D_{X_A}\sigma$ is $R_A(s) + h_A s$. So the section $-D_{X_A}\sigma + h_A\sigma$ of \mathcal{V} has as associated Φ -equivariant function to $-R_A(s)$. Therefore if we put

$$\mathcal{P}_A(\sigma) := -D_{X_A}\sigma + h_A\sigma,$$

then the family $\{\mathcal{P}_A\}$ of endomorphisms is a representation of \mathfrak{g} on \mathcal{B}_2 equivalent to π' defined in (2.11).

From (3.10) it follows

$$Y_A[g, \alpha](\sigma^\sharp) = \alpha^{-1} R_A(g)(s).$$

That is, $Y_A \sigma^\sharp$ is the function of \mathcal{B}_3 associated to $R_A s \in \mathcal{B}_1$. Hence the algebra representation π' defined in (2.11) is equivalent to the representation of \mathfrak{g} on the space \mathcal{B}_3 given by the operators $\{-Y_A\}$. □

4. SCHUR'S LEMMA

Let $\{B_t\}_{t \in [0,1]}$ be a family of elements in \mathfrak{g} . This family generates time-dependent vector fields on G , \mathcal{O} and \mathcal{F} , which give rise to evolution equations for several sorts of objects. In Propositions 10, 11 and 12 we state properties of the solutions to these equations that we use later.

By φ_t we denote the isotopy on \mathcal{O} determined by the time-dependent vector field X_{B_t} ; that is,

$$(4.1) \quad \frac{d\varphi_t}{dt} = X_{B_t} \circ \varphi_t, \quad \varphi_0 = \text{id}.$$

On the other hand the time-dependent vector field Y_{B_t} on \mathcal{F} defines a flow \mathbf{H}_t ; that is, the family of diffeomorphisms of \mathcal{F} determined by

$$(4.2) \quad \frac{d\mathbf{H}_t(p)}{dt} = Y_{B_t}(\mathbf{H}_t(p)), \quad \mathbf{H}_0 = \text{Id}.$$

Given $s \in \mathcal{B}_1$, we define a family of maps $\{s_t : G \rightarrow H\}_t$ by the equations

$$(4.3) \quad \frac{ds_t}{dt} = -R_{B_t}s_t, \quad s_0 = s.$$

Proposition 10. *If the family $\{s_t\}$ is solution of (4.3), then $s_t \in \mathcal{B}_1$ for all t .*

Proof.

$$(4.4) \quad \frac{d}{dt}(L_Z s_t) = L_Z\left(\frac{ds_t}{dt}\right) = -L_Z R_{B_t}(s_t) = -R_{B_t} L_Z(s_t).$$

If $Z \in \mathfrak{u}$, then $L_Z s_0 = 0$. The uniqueness of solutions of the first order differential equation (4.4) implies $L_Z s_t = 0$, for all t .

On the other hand, given $l \in L$, on the space of smooth maps $h : G \rightarrow H$ we define the operator α by

$$\alpha(h) = h \circ R_l - \Phi(l^{-1})h,$$

R_l being the right multiplication in G by l . If $A \in \mathfrak{g}$, it is straightforward to check

$$(4.5) \quad \alpha R_A = R_A \alpha.$$

If s_t is solution of (4.3), it follows from (4.5)

$$\frac{d}{dt} \alpha(s_t) = \frac{d s_t}{dt} \circ R_l - \Phi(l^{-1}) \frac{d s_t}{dt} = \alpha(-R_{B_t} s_t) = -R_{B_t}(\alpha s_t).$$

As $\alpha(s) = 0$ since $s \in \mathcal{B}_1$, so we conclude $\alpha s_t = 0$ for all t . Thus $s_t \in \mathcal{B}_1$. \square

Given a section $\sigma \in \mathcal{B}_2$ we can consider the family of sections σ_t of \mathcal{V} defined by the following equations

$$(4.6) \quad \frac{d \sigma_t}{dt} = \mathcal{P}_{B_t} \sigma_t, \quad \sigma_0 = \sigma.$$

Similarly, given $f \in \mathcal{B}_3$, let f_t be the set of functions $f_t : \mathcal{F} \rightarrow H$ such that

$$(4.7) \quad \frac{d f_t(p)}{dt} = -Y_{B_t}(p)(f_t), \quad f_0 = f.$$

By Theorem 1 together with the preceding Proposition one has

Proposition 11. *Let σ_t be the solution of (4.6) and f_t the solution of (4.7). If σ and f are associated with $s \in \mathcal{B}_1$, then σ_t and f_t are associated with s_t , solution of (4.3). In particular $\sigma_t \in \mathcal{B}_2$ and $f_t \in \mathcal{B}_3$.*

Given a $f \in \mathcal{B}_3$, we put

$$(4.8) \quad \hat{f}_t = f \circ \mathbf{H}_t^{-1}.$$

We have the following proposition.

Proposition 12. *The set of functions \hat{f}_t defined by (4.8) satisfies*

$$\frac{d \hat{f}_t(p)}{dt} = -Y_{B_t}(p)(\hat{f}_t), \quad \hat{f}_0 = f.$$

Hence \hat{f}_t is the solution of (4.7).

Proof. By (4.2)

$$(4.9) \quad \left. \frac{d}{du} \right|_{u=t} \mathbf{H}_u(\mathbf{H}_t^{-1}(p)) = Y_{B_t}(p);$$

that is, $Y_{B_t}(p)$ is the vector defined by the curve $\{\mathbf{H}_u(\mathbf{H}_t^{-1}(p))\}_u$ at $u = t$. On the other hand, by (4.8)

$$\frac{d \hat{f}_t(p)}{dt} = W(f),$$

where W is the tangent vector to \mathcal{F} at $\mathbf{H}_t^{-1}(p)$ defined by the curve $\{\mathbf{H}_u^{-1}(p)\}_u$. Since $\mathbf{H}_u(\mathbf{H}_u^{-1}(p)) = p$ for all u , it turns out that $Y_{B_t}(p) = -(\mathbf{H}_t)_*(W)$. So

$$-Y_{B_t}(p)(\hat{f}_t) = W(\hat{f}_t \circ \mathbf{H}_t) = W(f) = \frac{d \hat{f}_t(p)}{dt}.$$

\square

If we integrate the family $\{B_t\}$ we will obtain the solutions of differential equations (4.1) and (4.2). That is, we define the curve b_t in G by the conditions

$$(4.10) \quad \dot{b}_t b_t^{-1} = B_t, \quad b_0 = e.$$

Then the isotopy φ_t determined by (4.1) is the multiplication by b_t ; that is,

$$(4.11) \quad \varphi_t(g \cdot \eta) = b_t \cdot (g \cdot \eta).$$

Analogously, the bundle diffeomorphism \mathbf{H}_t defined in (4.2) is the left multiplication by b_t in \mathcal{F} ,

$$(4.12) \quad \mathbf{H}_t = \mathcal{L}_{b_t}.$$

Proposition 13. *The solution of (4.3) is $s_t = s \circ L_{b_t^{-1}}$, where b_t is determined by (4.10).*

Proof. We write $\tilde{s}_t := s \circ L_{b_t^{-1}}$. Given $g \in G$

$$\frac{d\tilde{s}_t}{dt}(g) = \frac{d}{du} \Big|_{u=t} s(b_u^{-1}g).$$

But the curves $\{b_u^{-1}\}_u$ and $\{b_t^{-1}b_u b_t^{-1}\}_u$ in the group G define opposite tangent vectors at $u = t$. So

$$\frac{d\tilde{s}_t}{dt}(g) = -\frac{d}{du} \Big|_{u=t} s(b_t^{-1}b_u b_t^{-1}g).$$

On the other hand,

$$R_{A_t}(\tilde{s}_t)(g) = \frac{d}{du} \Big|_{u=t} \tilde{s}_t(b_u b_t^{-1}g) = \frac{d}{du} \Big|_{u=t} s(b_t^{-1}b_u b_t^{-1}g).$$

That is, $s \circ L_{b_t^{-1}}$ satisfies (4.3). \square

Proof of Theorem 2. Now $g_1 \in Z(G)$ and the isotopy ψ_t defined in (1.5) is closed; that is, $\psi_1 = \text{id}$. By Proposition 13, (2.10) and (1.1) one has

$$s_1 = s \circ L_{g_1^{-1}} = \pi_1(g_1)(s) = \kappa s,$$

for any $s \in \mathcal{B}_1$, which proves item (a).

By (3.9) and Proposition 11, the result stated in (a) expressed in terms of the solutions to (1.3) gives (b).

Moreover

$$\kappa s(g) = s_1(g) = s(g_1^{-1}g) = s(gg_1^{-1}) = \Phi(g_1)s(g);$$

that is,

$$(4.13) \quad \Phi(g_1) = \kappa \text{Id}.$$

It follows from (4.12) and (4.13) that

$$(4.14) \quad \mathbf{F}_1[g, \alpha] = [g_1 g, \alpha] = [gg_1, \alpha] = [g, \Phi(g_1)\alpha] = \kappa[g, \alpha],$$

and (d) is proved.

From Proposition 12 together with (4.12) and (4.14), it follows

$$f_1[g, \alpha] = (f \circ \mathbf{F}_1^{-1})[g, \alpha] = f[g, \kappa^{-1}\alpha] = \kappa f[g, \alpha],$$

which proves (c). \square

Let us assume that there is a fixed point $x_0 \in \mathcal{O}$ for the isotopy $\{\psi_t\}$ defined in (1.5); that is, $\psi_t(x_0) = x_0$, for all t . So $X_{A_t}(x_0) = 0$ and (1.3) evaluated at x_0 reduces to

$$\frac{d\sigma_t(x_0)}{dt} = h_{A_t}(x_0)\sigma_t(x_0), \quad \sigma_0(x_0) = \sigma(x_0).$$

This is a differential linear equation for $v_t := \sigma_t(x_0) \in H$. Let $M(t) \in \mathfrak{gl}(H)$ be the “fundamental matrix” of this linear equation, in other words

$$\frac{dM(t)}{dt} = h_{A_t}(x_0)M(t), \quad M(0) = \text{Id}.$$

By Theorem 2 it follows

$$(4.15) \quad M(1) = \kappa \text{Id}.$$

Corollary to Theorem 2. *If $A_t = A$ for all t , and x_0 is a fixed point of the isotopy $\{\psi_t\}_t$, then*

$$\kappa \text{Id} = \exp(h_A(x_0)).$$

4.1. Case when $\dim H = 1$. Now the bracket and the adjoint action in $\mathfrak{gl}(H)$ are trivial. It follows from (3.7)

$$\mathbf{K}_{[g, \alpha]}(Y_B, Y_C) = -\phi(g^{-1}[B, C]) = -h_{[B, C]}(g),$$

and \mathbf{K} projects a closed 2-form K on \mathcal{O} . We denote by $\omega := -K$; that is,

$$(4.16) \quad \omega(X_A, X_B)(g \cdot \eta) = h_{[A, B]}(g).$$

In this case (3.8) reduces to

$$(4.17) \quad dh_B = \iota_{X_B}\omega.$$

In this context ψ_t defined in (1.5) is an isotopy which determines the time-dependent Hamiltonian h_{A_t} through the form ω [9].

Proof of Theorem 3. Let μ be a local frame for the line bundle \mathcal{V} . The solution σ_t to (1.3) can be written $\sigma_t = m_t\mu$, where m_t is a complex function defined on an open set of \mathcal{O} . Then (1.3) gives rise to

$$(4.18) \quad \frac{dm_t}{dt} = -\gamma(X_{A_t})m_t - X_{A_t}(m_t) + h_{A_t}m_t,$$

where γ is the connection form of \mathcal{V} in the frame μ . Given an arbitrary point q of \mathcal{O} , then $\{q_t := \psi_t(q)\}_t$ is a closed curve on \mathcal{O} . If q belongs to the domain of μ , we define $m'_t := m_t(q_t)$. If we evaluate (4.18) at the point q_t we obtain

$$\frac{dm'_t}{dt} = (-\gamma_{q_t}(X_{A_t}) + h_{A_t}(q_t))m'_t.$$

So

$$m'_t = m'_0 \exp\left(\int_0^t (-\gamma_{q(u)}(X_{A_u}) + h_{A_u}(q_u))du\right).$$

On the other hand, we can consider on \mathcal{O} the Kirillov symplectic structure [2], then ψ_t is a Hamiltonian isotopy with respect to this structure, and consequently the evaluation closed curve $\{q_t\}$ is nullhomologous (Lemma 10.31 in [9]), that is, it is the boundary of a 2-chain. By Stokes' theorem

$$m'_1 = m'_0 \exp\left(\int_S \omega + \int_0^1 h_{A_t}(q_t)dt\right),$$

where S a 2-chain whose boundary is the curve is $\{q_t\}_t$. By Theorem 2

$$(4.19) \quad \kappa = \exp\left(\int_S \omega + \int_0^1 h_{A_t}(q_t) dt\right).$$

□

Remarks. If q is a fixed point for ψ_t and $A_t = A$ for all t , from Theorem 3 it follows $\kappa = \exp(h_A(q))$; this agrees with Corollary to Theorem 2.

By (1.1) the exponential in the statement of Theorem 3 depends only on the final point of the curve g_t and it is independent of the family A_t defined by g_t .

4.2. The character. A slight modification of the preceding developments allows us to prove the formula for the character given in Theorem 4.

Proof of Theorem 4. Let c_t denote a path in L with $c_1 = a^{-1}\tilde{g}a$. Then $g_t := ac_t a^{-1}$ is a path in G with $g_1 = \tilde{g}$. The point $x_0 = a \cdot \eta \in \mathcal{O}$ is a fixed point for the isotopy on \mathcal{O} defined by multiplication by g_t . We put $A_t \in \mathfrak{g}$ and $C_t \in \mathfrak{l}$ for the velocity paths associated with g_t and c_t , respectively. So $A_t = \text{Ad}_a C_t$.

We denote by V_2 the subspace of \mathcal{B}_2 on which τ is defined. The action of $\tau(\tilde{g})$ on a section $\sigma \in V_2$ is the section $\sigma(1)$ determined by the equations

$$\frac{d\sigma(t)}{dt} = \mathcal{P}_{A_t}(\sigma(t)), \quad \sigma(0) = \sigma.$$

Evaluating these equations at the point x_0 , and taking into account that $X_{A_t}(x_0) = 0$, one obtains

$$(4.20) \quad \frac{d\sigma(t)(x_0)}{dt} = h_{A_t}(x_0)(\sigma(t)(x_0)), \quad \sigma(0)(x_0) = \sigma(x_0).$$

As $\dim H = 1$, it follows from (4.20) that

$$(4.21) \quad \sigma(1)(x_0) = \exp\left(\int_0^1 h_{A_t}(x_0) dt\right) \sigma(x_0),$$

for any $\sigma \in V_2$.

If $\dim \tau = m$, let $\sigma_1, \dots, \sigma_m$ be a basis of V_2 , then

$$\sigma_j(1) = \tau(\tilde{g})\sigma_j = \sum_i M_{ij}\sigma_i,$$

with $M_{ij} \in \mathbb{C}$. By (4.21)

$$M_{ji} = \delta_{ji} \exp\left(\int_0^1 h_{A_t}(x_0) dt\right),$$

and the proof is complete. □

Remark. As $h_{A_t}(x_0) = \phi(a^{-1} \cdot A_t) = \phi(C_t)$ the formula for the character can be written

$$\chi_\tau(\tilde{g}) = m \exp\left(\phi\left(\int_0^1 C_t dt\right)\right).$$

If $\tilde{c} := a^{-1}\tilde{g}a$ equals e^C , with $C \in \mathfrak{l}$, then we can take $C_t = C$ for all t and

$$(4.22) \quad \chi_\tau(\tilde{g}) = m \Phi(\tilde{c}).$$

(4.22) can also be deduced by considering τ as a representation on a subspace V_1 of \mathcal{B}_1 . Given $s \in V_1$,

$$(\tau(\tilde{g})s)(a) = s(a\tilde{c}^{-1}) = \Phi(\tilde{c})s(a).$$

From this formula it follows (4.22).

4.3. The infinitesimal character. Now we consider the “representation” of the associative algebra $U(\mathfrak{g}_{\mathbb{C}})$ induced by π' on the space $(\mathcal{B}_2)_K$ of K -finite vectors in \mathcal{B}_1 . Since π is unitary and irreducible, each element of $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ acts as a scalar operator (see [4], Corollary 8.14). By $\chi : \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \rightarrow \mathbb{C}$ we denote the corresponding infinitesimal character.

Let \mathfrak{h} be a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ contained in $\mathfrak{l}_{\mathbb{C}}$. We denote by Δ^+ a set of positive roots for the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$. For $\alpha \in \Delta^+$, E_{α} will be a basis for the corresponding root space. According to Lemma 8.17 in [4], if $Z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ then $Z \in U(\mathfrak{h}) \oplus \mathcal{P}$, where

$$\mathcal{P} = \sum_{\alpha \in \Delta^+} U(\mathfrak{g}_{\mathbb{C}})E_{\alpha}.$$

The projection of Z into $U(\mathfrak{h})$ will be denoted \hat{Z} .

Let $V \subset (\mathcal{B}_2)_K$ be an irreducible representation of the maximal compact subgroup K which occurs in $\pi|_K$. Now we consider $s_0 \in V$ a highest weight vector of the representation V , so $\pi'(E_{\alpha})s_0 = 0$ for all $\alpha \in \Delta^+$. Thus, if $Z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \cap \mathcal{P}$ then the action of Z on s_0 vanishes. As $\pi'(Z)s_0 = \chi(Z)s_0$, it follows $\chi(Z) = 0$. We have the following proposition

Proposition 14. *If $Z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \cap \mathcal{P}$, then $\chi(Z) = 0$.*

To prove Theorem 5 we need the following Lemma

Lemma 15. *With ϕ denoting the extension of the map (2.7) to $\mathfrak{g}_{\mathbb{C}}$ and 1 the identity element of G , we have*

- (i) *If $Y \in \mathfrak{h}$ and $s \in \mathcal{B}_1$, then $\pi'(Y)s = \phi(Y)s$.*
- (ii) *If $Y, W \in \mathfrak{h}$, then*

$$(\pi'(Y)\pi'(W)s)(1) = \phi(Y)\phi(W)s(1)$$

Proof. Since $\mathfrak{h} \subset \mathfrak{l}_{\mathbb{C}}$ we can assume that Y, W are elements of \mathfrak{l} . The item (i) follows from (2.11) together with the fact that s is Φ -equivariant.

If $Y, W \in \mathfrak{l}$, then $s(e^{uW}e^{tY}) = \Phi(e^{-tY})\Phi(e^{-uW})s(1)$. Hence

$$(R_Y R_W s)(1) = \frac{d}{dt} \Big|_{t=0} \frac{d}{du} \Big|_{u=0} \Phi(e^{-tY})\Phi(e^{-uW})s(1) = \phi(Y)\phi(W)s(1),$$

and (ii) follows. \square

Given $\{Y_1, \dots, Y_r\}$ a basis of \mathfrak{h} , and $Z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$, then there exist a polynomial $q(Y_1, \dots, Y_r)$ as in (1.6) such that $\hat{Z} = q(Y_1, \dots, Y_r)$.

Proposition 16. *With the above notations*

$$q(\phi(Y_1), \dots, \phi(Y_r)) = \chi(Z) \text{Id}.$$

Proof. By Proposition 14, the operator $\pi'(Z)$ associated to Z is $q(-R_{Y_1}, \dots, -R_{Y_r})$. By Lemma 15, if $s \in \mathcal{B}_1$ then

$$(q(-R_{Y_1}, \dots, -R_{Y_r})s)(1) = q(\phi(Y_1), \dots, \phi(Y_r))s(1).$$

As $q(-R_{Y_1}, \dots, -R_{Y_r}) = \chi(Z) \text{Id}$, we obtain the proposition. \square

Theorem 17. *Given $Z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$, if $\hat{Z} = q(Y_1, \dots, Y_r)$ and $h_k := h_{Y_k}$, the Hamiltonian map associated with Y_k , then the function*

$$q(h_1, \dots, h_r) : \mathcal{O}_{\eta} \rightarrow \mathfrak{gl}(H)$$

takes at the point η the value $\chi(Z)\text{Id}$.

Proof. It follows from Proposition 16 and (3.1). □

Proof of Theorem 5. If \tilde{q} is any polynomial in the variables Y_1, \dots, Y_r , from (3.5) one deduces that

$$(4.23) \quad \tilde{q}(\mathbf{h}_1, \dots, \mathbf{h}_r)[g, \alpha] = \alpha^{-1}(\tilde{q}(h_1, \dots, h_r)(g \cdot \eta))\alpha.$$

On the other hand, $q(h_1, \dots, h_r)(\eta)$ is a multiple of identity, by Theorem 17. From this fact together with (4.23) we deduce

$$q(\mathbf{h}_1, \dots, \mathbf{h}_r)[1, \alpha] = q(h_1, \dots, h_r)(\eta),$$

for any $[1, \alpha] \in \mathcal{F}$. The theorem follows from Theorem 17. □

Let $x_0 = g \cdot \eta$ be a point of \mathcal{O} and let \mathfrak{h}' be a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ such that, $g^{-1} \cdot \mathfrak{h}' \subset \mathfrak{l}_{\mathbb{C}}$. A generalization of Theorem 5 is the following proposition

Proposition 18. *If Y'_1, \dots, Y'_r is a basis of \mathfrak{h}' and the polynomial $q'(Y'_1, \dots, Y'_r)$ is the projection of $Z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ into $U(\mathfrak{h}')$, then the function $q'(\mathbf{h}_{Y'_1}, \dots, \mathbf{h}_{Y'_r})$ is constant on the fiber of \mathcal{F} over x_0 , and its value on this fiber is $\chi(Z)\text{Id}$.*

5. PHYSICAL INTERPRETATIONS

As it is well known Geometric Quantization [17] is a mathematical procedure for understanding the relation between a classical physical system and its “quantization”. From the mathematical point of view the classical phase space is a symplectic manifold (M, α) , and the set of rays of a Hilbert space \mathcal{H} is the mathematical model for the space of states of the quantum system. The manifold (M, α) is said to be quantizable if the cohomology class of $\alpha/(2\pi)$ is integral. In this case there exists a Hermitian line bundle \mathcal{L} on M equipped with a connection whose curvature is $-i\alpha$. \mathcal{L} is called a “prequantum bundle”. For the construction of the Hilbert space \mathcal{H} from the quantizable manifold M one fixes a polarization \mathfrak{P} on M , then \mathcal{H} is a subset of the space of sections of \mathcal{L} polarized with respect to \mathfrak{P} (see [17] and [10] for the details omitted in this schematic summary).

The coadjoint orbit \mathcal{O} of $\eta \in \mathfrak{g}^*$ supports a canonical symplectic structure $\hat{\omega}$, the Kirillov form [2]. Denoting by L the stabilizer of η , the orbit \mathcal{O} admits a G -invariant prequantization iff the linear operator $i\eta : \mathfrak{l} = \text{Lie}(L) \rightarrow i\mathbb{R}$ is integral, in the sense that there exists a character $\Lambda : L \rightarrow U(1)$ whose derivative is $i\eta$ [8]. In this case the corresponding prequantum bundle is $\mathcal{L} = G \times_{\Lambda} \mathbb{C}$. Since the group G acts by translation on the orbit, it is reasonable to impose the quantization to have a G -invariant Hilbert space structure. In general it is not possible to integrate the absolute value of sections of \mathcal{L} in a translation-invariant way, since $\mathcal{O} = G/L$ does not admit a measure invariant under the action of G (see p. 537 [5]). To define such an integration it is necessary to consider a prequantum bundle different from \mathcal{L} ; specifically, one takes the bundle \mathcal{V} determined by the character $\Phi = \Lambda \cdot \Delta$, where Δ^2 is the modular function on G/L . If σ_1, σ_2 are compactly supported

sections of \mathcal{V} , and $\underline{s_1, s_2} : G \rightarrow \mathbb{C}$ are the corresponding Φ -equivariant functions, then $m(g) := s_1(g)\overline{s_2(g)}$ satisfies

$$(5.1) \quad m(gl) = \Delta^2(l^{-1})m(g), \quad \text{for all } l \in L.$$

(That is, m defines a section of the bundle of densities on G/L .) The functions on G which satisfies (5.1) have a translation invariant integral “over G/L ” (see p. 65 [12], p. 41 [6]). Then $\langle \sigma_1, \sigma_2 \rangle := \int_{G/L} m$ defines a G -invariant product of compactly supported sections of \mathcal{V} .

On the other hand, if η is a hyperbolic element then the orbit \mathcal{O} possesses the polarization determined by the subalgebra \mathfrak{u} defined in (2.2). So our space \mathcal{B}_2 , defined in (3.15), is a G -invariant quantization of the orbit. By Proposition 9 the spaces \mathcal{B}_j , $j = 1, 2, 3$, can be considered as equivalent G -invariant quantizations of \mathcal{O} .

The Kirillov symplectic form $\hat{\omega}$ is defined by

$$\hat{\omega}_{g \cdot \eta}(X_A, X_B) = \eta(g^{-1} \cdot [A, B]),$$

and the Hamiltonian function associated to A is $\hat{h}_A(g \cdot \eta) = \eta(g^{-1} \cdot A)$. From (3.1), (4.16) and (2.7) one obtains

$$\omega = i\hat{\omega} + \tilde{\omega}, \quad h_A = i\hat{h}_A + \tilde{h}_A,$$

where

$$\tilde{\omega}_{g \cdot \eta}(X_A, X_B) = \delta(g^{-1} \cdot [A, B]), \quad \tilde{h}_A(g \cdot \eta) = \delta(g^{-1} \cdot A).$$

$\tilde{\omega}$ is not a symplectic form (because it is degenerate), but the analogous relations to (4.17) with tildes and with hats are also valid.

The cotangent bundle $M = T^*P$ to a manifold carries a canonical 1-form β_0 [9], and $\alpha_0 := -d\beta_0$ defines a symplectic structure on M . If $q : t \in [0, 1] \rightarrow M$ is a curve and $h_t : M \rightarrow \mathbb{R}$ is a time dependent Hamiltonian on M , the action integral along the curve $q(t)$ is defined by the following formula [1], [9]

$$\int_0^1 (-\beta_0(\dot{q}(t)) + h_t(q(t)))dt.$$

For a general symplectic manifold (M, α) the time dependent Hamiltonian h_t , with $t \in [0, 1]$, determines a time dependent Hamiltonian vector field X_t , which in turn defines an isotopy of symplectomorphisms ξ_t . If $\xi_1 = \text{id}$ (that is, $\{\xi_t\}_{t \in [0, 1]}$ is a loop in $\text{Ham}(M)$, the Hamiltonian group of M [9]), then evaluation curve $\{\xi_t(p)\}_t$ is nullhomotopic, for all point $p \in M$ [7]. Hence the action integral around this curve can be written

$$(5.2) \quad \hat{A}(\xi) := \int_S \alpha + \int_0^1 h_t(\xi_t(p))dt,$$

S being a 2-chain whose boundary is the curve $\{\xi_t(p)\}_t$. It is known that the value of (5.2) is independent of the point p [11].

In the case when the manifold is a coadjoint orbit \mathcal{O} and the loop $\{\psi_t\}$ in $\text{Ham}(\mathcal{O})$ is defined as in (1.5), one can also consider the “action integral” $\tilde{\mathcal{A}}(\psi)$ defined by means of the 2-form $\tilde{\omega}$ and the “Hamiltonian” \tilde{h}_{A_t}

$$\tilde{\mathcal{A}}(\psi) := \int_S \tilde{\omega} + \int_0^1 \tilde{h}_{A_t}(\psi_t(p))dt.$$

Thus the result stated in Theorem 3 can be written as $\kappa = \exp(i\hat{A}(\psi)) \times \exp(\tilde{A}(\psi))$. Since the representation π is unitary, $\kappa \in U(1)$. So

$$\kappa = \exp(i\hat{A}(\psi)).$$

That is, the invariant of the representation π associated to $g_1 \in Z(G)$, by Schur's lemma, equals the exponential of i times the action integral around the loop in $\text{Ham}(\mathcal{O})$ generated by any path in G with endpoint at g_1 .

In view of item (d) of Theorem 2 and the above facts, we may consider the flow \mathbf{H}_t (defined in (4.2)) as a generalized action integral along the isotopy φ_t , relative to the integral datum Λ .

The Berry phase is a general phenomenon which may appear when a quantum system undergoes a cyclic evolution. We summarize the geometric definition of Berry phase given in [16], where general references can be find. Let $\{(N_t, \epsilon_t)\}_{t \in [0,1]}$ be a family of weighted Lagrangian submanifolds of (M, α) (ϵ_t being a smooth density on N_t), obtained from $N = N_0$ by a set ξ_t of symplectomorphisms of M determined by a time dependent Hamiltonian h_t . Let us assume that (M, α) is quantizable and \mathcal{L} is a prequantum bundle. $\mathcal{L}^\times = \mathcal{L} \setminus \{\text{zero section}\}$ is the corresponding principal bundle. We denote by F_t the flow on \mathcal{L}^\times generated by the vector field $X_t^\sharp - W_{h_t}$, where X_t^\sharp is the horizontal lift of the respective Hamiltonian vector field X_t , and W_{h_t} is h_t times the fundamental vertical vector field on \mathcal{L}^\times . If σ is a section of \mathcal{L}_N^\times , then $F_1(\sigma(N))$ differs from $\sigma(N)$ by a phase θ . If the Hamiltonian h_t are normalized so that $\int_{N_t} h_t \epsilon_t = 0$, then θ is the Berry phase of the loop $\{(N_t, \epsilon_t)\}_{t \in [0,1]}$ (p.142 [16]).

By (3.17) the statement (d) in Theorem 2, when $\dim H = 1$, can be interpreted by saying that the invariant κ is the Berry phase of any loop $\{(N_t, \epsilon_t)\}_{t \in [0,1]}$ generated by the Hamiltonian functions h_{A_t} , where A_t is the velocity curve of any path in G with endpoint at g_1 , and h_{A_t} is given by (3.1).

Example. Let G be the restricted Lorentz group $SO^+(1, 3)$. A basis for the Lie algebra $\mathfrak{so}^+(1, 3)$ is X_1, \dots, X_6 , where X_1, X_2, X_3 are generators of the boosts along the axes, and X_4, X_5, X_6 are the generators of the rotations around those axes. The matrix of Killing metric in the basis X_i is

$$(g_{ij}) = (\text{Tr}(\text{ad } X_i \circ \text{ad } X_j)) = \text{diag}(1, 1, 1, -1, -1, -1),$$

and the Casimir element C of $U(\mathfrak{g}_\mathbb{C})$ is $C = \sum g_{ij} X^i Y^j$, where $X^i = \sum_k g^{ia} X_k$ [4]. That is,

$$(5.3) \quad C = \frac{1}{4} \left(\sum_1^3 X_i^2 - \sum_4^6 X_i^2 \right).$$

Let $Y = (Y_{ab})$ be a matrix in $\mathfrak{so}^+(1, 3)$, with $a, b = 0, 1, 2, 3$, and let η denote the element in $\mathfrak{so}^+(1, 3)^*$ defined by $\eta(Y) = kY_{01}$, with $k \in \mathbb{R} \setminus \{0\}$ (equivalently $\eta(X_j) = k\delta_{1j}$). The matrix associated with η according to (2.1) is precisely kX_1 , which has real eigenvalues; i.e. the coadjoint orbit \mathcal{O} of η is hyperbolic. Furthermore \mathcal{O} is G/L , with $L = SO^+(1, 1) \times SO(2)$. If $(A, B) \in L$, then A will have the form $A = \exp aX_1$ and η can be extended to a character Λ on L by putting $\Lambda(A, B) = e^{ika}$.

One has the following relations

$$[X_1, X_2] = X_6, \quad [X_1, X_3] = X_5, \quad [X_1, X_4] = 0, \quad [X_1, X_5] = X_3, \quad [X_1, X_6] = X_2.$$

So a basis for the subalgebra \mathfrak{u} defined in (2.2) is $X_2 + X_6$, $X_3 + X_5$, and for the operators δ and ϕ introduced in Section 2 we have

$$(5.4) \quad \delta(X_1) = \text{Tr}(\text{ad}(X_1)|_{\mathfrak{u}}) = 2, \quad \phi(X_1) = ik + 2.$$

Analogously

$$(5.5) \quad \delta(X_4) = 0, \quad \phi(X_4) = 0.$$

$\mathfrak{h} = \mathfrak{l}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$, and it is generated by X_1, X_4 . By (5.3), the projection \hat{C} of C on $U(\mathfrak{h})$ is $\hat{C} = \frac{1}{4}(X_1^2 - X_4^2)$. By Theorem 5, it follows from (5.4), (5.5) and (3.1) that the value $\chi(C)$ of the infinitesimal character of the representation π' associated with the orbit \mathcal{O} is $(1/4)(ik + 2)^2$.

For $A \in \mathfrak{g}$, the operator $\mathcal{P}_A = -D_{X_A} + h_A$ acting on polarized sections of the prequantum bundle is the “quantization” of the vector field on \mathcal{O} determined by A [10]. From the above result it turns out that the operator

$$\frac{1}{4} \left(\sum_{i=1}^3 (\mathcal{P}_{X_i})^2 - \sum_{i=4}^6 (\mathcal{P}_{X_i})^2 \right),$$

associated with the Casimir element $C \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ is simply the multiplication by the constant $(1/4)(ik + 2)^2$.

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